MR8500 - PhD Topics in Marine Control Systems (2020)

Backstepping design on complex nonlinear ODE systems

Lecture 1: Elegant methods

Lecture 2: Transferring a complex system into a familiar form

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Nov. 19, 2020

Some examples about backstepping designs of complex nonlinear system

Journals & Magazines > IEEE Transactions on Fuzzy Sy... > Volume: 20 Issue: 1

Adaptive Fuzzy Output Feedback Tracking Backstepping Control of Strict-Feedback Nonlinear Systems With Unknown Dead Zones

Journals & Magazines > IEEE Transactions on Systems,... > Volume: 34 Issue: 1

Adaptive neural control of nonlinear time-delay systems with unknown virtual control coefficients

ELSEVIER

Neurocomputing

Volume 175, Part A, 29 January 2016, Pages 759-767

Adaptive backstepping-based fuzzy tracking control scheme for output-constrained nonlinear switched lower triangular systems with time-delays ☆

ELSEVIER

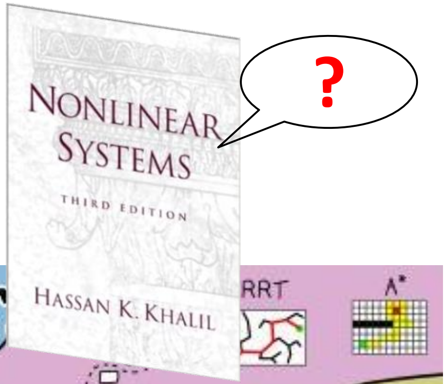
Automatica

Volume 64, February 2016, Pages 70-75

Brief paper

Barrier Lyapunov Functions-based adaptive control for a class of nonlinear pure-feedback systems with full state constraints ☆

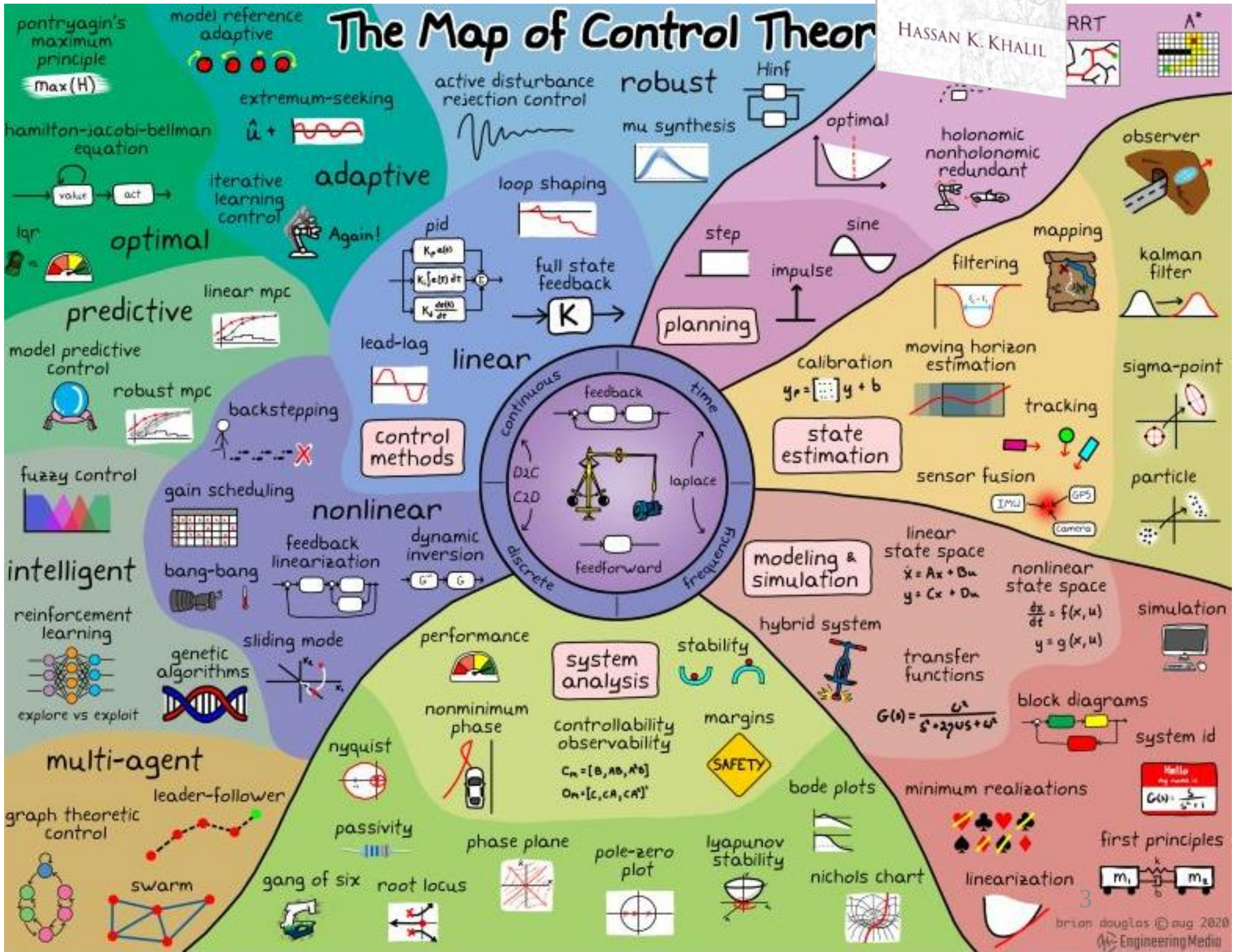
Beyond the textbook



Backstepping is symmetric, recursive, and Lyapunov-based design.

However, the scope of control theories is broad and heterogeneous.

How to apply backstepping to more complex nonlinear systems?



Backstepping is similar to cook fast food

Outline

Lecture 1 - Elegant methods

- The development of backstepping, from simple systems to complex uncertain systems.
- Semi-global stability criteria
- Six modularizable methods
  - Dynamic surface control / commanded filters
  - Finite-time control
  - Neural network and fuzzy logic system
  - Nussbaum function
  - Barrier Lyapunov function
  - Hyperbolic tangent function.

Selection standards

- Widely-used
- Easy to use
- Modularizable
- Compatible with other methods

Lecture 2 - Applications of methods in Lecture 1 to complex nonlinear systems

- A class of systems:
  - State constraints
  - Input nonlinearities (input saturation, deadzone, time-varying control coefficient),
  - Unknown disturbance
  - Time-delay effects
  - Pure-feedback system
  - Event-triggered systems
  - Stochastic systems
- Complex systems:
  - Underactuated system
  - Switched system
  - Multi-agent consensus system.
- Understand the robustness-based method and the approximation-based method

Lecture 1-Elegant methods

1.0 Some notations

- Stabilization ($x_1 \to 0$)/Tracking ($x_1 \to x_{1d}$)
- State feedback ($u(x)$)/Output feedback ($u(\hat{x})$)
- Strictly feedback /Pure feedback (no explicit virtual control coefficient)
- Deterministic system/Stochastic system
- ODE/PDE
- SISO/MIMO
- Choose control gain in the order of deduction/presentation of results

You will only learn how to solve in the lectures.
(The question of "why" are left for interested readers.)

Some symbols
$\forall$ - For all/for every
$\exists$ - Exists
iff - If and only if
$\in$ - In
$\underline{a}$ - Lower limit of a
$\overline{a}$ - Upper limit of a
$\mathcal{I} = \{1,2,\cdots,n-1\}$

1.1 General design approaches

Quadratic Lyapunov function

$V_{i,QF} = \frac{1}{2} z_i^2$ and $\dot{V}_{i,QF} = z_i \dot{z}_i$

$$\dot{x}_i = f_i(\bar{x}_i) + g_i(\bar{x}_i) x_{i+1} + \phi_i(\bar{x}_i)^\top \theta_i, i \in \mathcal{I}$$

$$\dot{x}_n = f_n(\bar{x}_n) + g_n(\bar{x}_n) u + \phi_n(\bar{x}_n)^\top \theta_n,$$

$$y = x_1,$$

where

$x_1, \cdots, x_n$	States
u	Control input
y	Output
$\theta \in \mathbb{R}^p$	Unknown constant vector
$f_1, \cdots, f_n, g_1, \cdots, g_n$	Smooth functions
g_i	Control coefficient function
$\bar{x}_i = [x_1, x_2, \cdots, x_i]^\top$	State vector

Control objective: $x_1 \to x_{1d}$ for $t \to \infty$

Assumptions: (Very important!)

- $x_{1d}(t)$ and its derivatives up to the required number of order are known, bounded, and continuous.
- (1) The signs of $g_1, \cdots, g_n$ are assumed to be known and constant; (2) $g_i^{(j)}$ are known and bounded; (3) $|g_i| > 0$ for all t

When $f_1 = \cdots = f_n = 0$ and $g_1 = \cdots = g_n = 1$, the system is simplified to be an integrator chain, or, namely, the Brunovsky form.

Define

α_i	Virtual control law
$z_1 = x_1 - x_{1d}$	$z_{i+1} = x_{i+1} - \alpha_i$
$\bar{z}_i = [z_1, z_2, \cdots, z_i]^\top$	$\bar{z}_{i:j} = [z_i, z_{i+1}, \cdots, z_j]^\top$

(Adaptive) integrator backstepping

Step 1: (a) Define $\tilde{\theta}_i = \theta_i - \hat{\theta}_i$ and a Lyapunov function candidate (LFC)

$$V_1(z_1) = V_{QF,1} \boxed{+\frac{1}{2}\tilde{\theta}_1^\top \Gamma_1^{-1} \tilde{\theta}_1}. \quad (1)$$

(b) Because $\dot{\tilde{\theta}}_i = -\dot{\hat{\theta}}_i$, its time derivative becomes

$$\dot{V}_1 = z_1[f_1 + g_1(\alpha_1 + z_2) \boxed{+\phi_1^\top \theta} - \dot{x}_{1d}] \boxed{-\tilde{\theta}_1^\top \Gamma_1^{-1} \dot{\hat{\theta}}_1}, \quad (2)$$

(c) The virtual control law and adaptive law are selected as

$$\alpha_1 = g_1^{-1}[-f_1(x_1) - \kappa_1(z_1) + \dot{x}_{1d} \boxed{-\phi_1^\top \hat{\theta}_1}], \quad (3)$$

$$\dot{\hat{\theta}}_1 = \Gamma_1 \phi_1 z_1, \quad (4)$$

where $\kappa_1(z_1) z_1$ is positive definite and $\gamma_1 > 0$. A simple example of $\kappa_1(z_1)$ is $\kappa_1(z_1) = c_1 z_1$ with $c_1 > 0$.

Step i ($i = 2 \cdots n-1$): $V_i = V_{i-1} + V_{i,QF} \boxed{+\frac{1}{2}\tilde{\theta}_i^\top \Gamma_i^{-1} \tilde{\theta}_i}$

$$\dot{V}_i = -\sum_{k=1}^{i-1} z_k \kappa_k(z_k) + z_i[g_{i-1} z_{i-1} + f_i + g_i(z_{i+1} + \alpha_i) \boxed{+\phi_i^\top \theta_i} - \dot{\alpha}_{i-1}] \boxed{-\tilde{\theta}_i^\top \Gamma_i^{-1} \dot{\hat{\theta}}_i}$$

Step n: $V_n = V_{n-1} + V_{QF,n} \boxed{+\frac{1}{2}\tilde{\theta}_n^\top \Gamma_n^{-1} \tilde{\theta}_n}$

$$\dot{V}_n = -\sum_{k=1}^{n-1} z_k \kappa_k(z_k) + z_n[g_{n-1} z_{n-1} + f_n + g_n u \boxed{+\phi_i^\top \theta} - \dot{\alpha}_{n-1}] \boxed{-\tilde{\theta}_n^\top \Gamma_n^{-1} \dot{\hat{\theta}}_n}$$

1.2 Summary of basic backstepping control

$$\dot{x}_i = f_i(\bar{x}_i) + g_i(\bar{x}_i)x_{i+1} + \phi_i(\bar{x}_i)^\top \theta_i, i \in \mathcal{I}$$
$$\dot{x}_n = f_n(\bar{x}_n) + g_n(\bar{x}_n)u + \phi_n(\bar{x}_n)^\top \theta_n,$$
$$y = x_1,$$

Benefit:

Transfer a class of systems into a group of simple problems and solved it by a sequential superposition of the corresponding approaches for each problem.

Keywords of backstepping design:

- Recursive cancellation - However, we cannot ensure everything is well canceled in a practical application.
- Smooth system
- Strictly-feedback system

Remark:
Deduction is not feasible without the assumptions.

Another application of adaptive backstepping is model identification. If the library functions ϕ_i are well defined, the system model can be identified.

Two problems:

1. Overparameterization problem caused by $\dot{\hat{\theta}}_i$
2. "Explosion of complexity" problem caused by $\frac{d^k \alpha_i}{dt^k}$ $(\frac{d^k f_i}{dx_j}\dots)$

1.3 Adaptive backstepping control using tuning functions

To overcome overparameterization problem

$$\dot{x}_i = f_i(\bar{x}_i) + g_i(\bar{x}_i)x_{i+1} + \phi_i(\bar{x}_i)^\top \theta, i \in \mathcal{I}$$

$$\dot{x}_n = f_n(\bar{x}_n) + g_n(\bar{x}_n)u + \phi_n(\bar{x}_n)^\top \theta,$$

$$y = x_1,$$

Challenge: Overparameterization problem $\dot{\hat{\theta}}_1, \dot{\hat{\theta}}_2, \cdots, \dot{\hat{\theta}}_n$

Idea: Same candidate functions in all steps

Method: To estimate all the unknown parameters in Step n $(\dot{\hat{\theta}}_1, \dot{\hat{\theta}}_2, \cdots, \dot{\hat{\theta}}_n \to \dot{\hat{\theta}})$

Adaptive backstepping with tuning functions

Step 1: (a) Define a Lyapunov function candidate (LFC)

$$V_1(z_1) = V_{QF,1} + \frac{1}{2}\tilde{\theta}^\top \Gamma^{-1} \tilde{\theta}. \quad (1)$$

(b) Its time derivative becomes

$$\begin{aligned} \dot{V}_1 &= z_1[f_1 + g_1(\alpha_1 + z_2) + \phi_1^\top \theta - \dot{x}_{1d}] - \tilde{\theta}^\top \Gamma^{-1} \dot{\hat{\theta}} \\ &= z_1[f_1 + g_1(\alpha_1 + z_2) + \phi_1^\top \hat{\theta} - \dot{x}_{1d}] - \tilde{\theta}^\top \Gamma^{-1}(\dot{\hat{\theta}} - \Gamma \phi_1 z_1), \end{aligned} \quad (2)$$

(c) The virtual control law is selected as

$$\alpha_1 = g_1^{-1}[-\kappa_1(z_1) - f_1 + \dot{x}_{1d} - \phi_1^\top \hat{\theta}], \quad (3)$$

Tuning function

$$\tau_1 = \gamma_1 z_1 \phi_1, \quad (4)$$

Substituting the virtual control law to the LFC and error dynamics yields

$$\dot{V}_1 = -\kappa(z_1)z_1 + z_1 g_1 z_2 + \tilde{\theta}^\top \Gamma^{-1}(\dot{\hat{\theta}} - \tau_1), \quad (5)$$

$$\dot{z}_1 = -\kappa_1(z_1)z_1 + z_2 + \phi_1^\top \tilde{\theta}, \quad (6)$$

Step i $(i = 2 \cdots n-1)$: $V_i = V_{i-1} + V_{i,QF}$

$$\dot{V}_i = -\sum_{k=1}^{i-1} z_k^\top \kappa_k(z_k) + z_i[g_{i-1}z_{i-1} + f_i + g_i(z_{i+1} + \alpha_i) - \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k}(f_k + g_k x_{k+1}) - \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}}\dot{\hat{\theta}}$$

$$+ \hat{\theta}^\top(\phi_i - \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k}\phi_k)] - \tilde{\theta}^\top \Gamma^{-1}\left(\dot{\hat{\theta}} - \tau_{i-1} - \Gamma z_i(\phi_i - \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k}\phi_k)\right)$$

$$\tau_i = \tau_{i-1} + \Gamma z_j(\phi_j - \sum_{k=1}^{j-1} \frac{\partial \alpha_{k-1}}{\partial x_k}\phi_k)$$

Step n: $V_n = V_{n-1} + V_{QF,n}$

$$u = g_n^{-1}[-\kappa_i(\bar{x}_n) - g_{n-1}z_{n-1} - f_n + \textstyle\sum_{k=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_k}(f_k + g_k x_{k+1}) + \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}}\dot{\hat{\theta}} - \hat{\theta}^\top(\phi_n - \sum_{k=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_k}\phi_k)]$$

$$\dot{\hat{\theta}} = \tau_n$$

A simpler method to understand this design is

- Cancel the unknown $\phi_i^\top \hat{\theta}$ in the virtual control
- Design the adaptive update only in the final step using $V = \sum_i^n V_i + \frac{1}{2}\tilde{\theta}^\top \Gamma^{-1} \tilde{\theta}$

Weak robustness property to non-parametric* uncertainty.
(depends on the selection of the library function $\varphi_1, \ldots, \varphi_n$)
* *Parametric*: for example $y = ax + bx^2$

M Krstic, I Kanellakopoulos, and PV Kokotovi´c. Adaptive nonlinear control without overparametrization. Systems & Control Letters, 19(3):177-185, 1992

2.0 The most important things in backstepping design beyond the former example
- Semi-global stability criteria & Young's inequality

Lemma 1. *A LFC $V(x)$ is bounded if the initial condition $V(0)$ is bounded, $V(x)$ is positive definite and continuous and if a Lyapunov-like inequality holds, i.e.,*

$$\dot{V}(x) \leq -\gamma V(x) + \delta, \tag{1}$$

where $\gamma > 0$ and $\delta > 0$. Define $\rho := \delta/\gamma$,

$$0 \leq V(t) \leq \rho + (V(0) - \rho)\exp(-\gamma t). \tag{2}$$

And it implies that

$$V(t) \leq e^{-\gamma t} V(0) + \int_0^t e^{-\gamma(t-\tau)} \rho(\tau) d\tau, \ \forall t \geq 0, \tag{3}$$

for any finite constant γ.

Some properties:
1. $V(z)$ outside the boundary goes into the boundary and stay in it after that.
2. If $V = \frac{1}{2}\sum_i^n z_i^2 \leq \rho$, $z_1 \leq \sqrt{2\rho}$. (Smaller δ and large $\gamma \Rightarrow$ smaller tracking error boundary)

Proof of Lemma 1. Times $\exp(\gamma t)$ to both sides of (1), yields

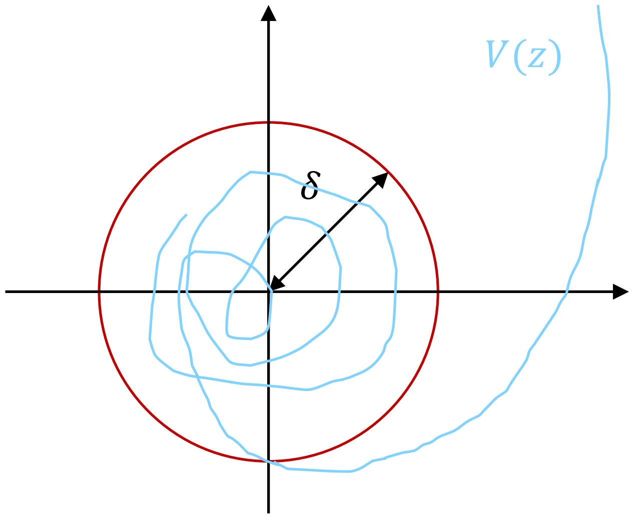
$$\dot{V}\exp(\gamma t) + \gamma V \exp(\gamma t) \leq \delta \exp(\gamma t).$$

$$\Rightarrow \frac{d}{dt} V \exp(\gamma t) \leq \frac{\delta}{\gamma}\frac{d}{dt}\exp(\gamma t).$$

Define $\rho := \delta/\gamma$. Integrating both side yields

$$V(t)\exp(\gamma t) - V(0) \leq \rho(\exp(\gamma t) - 1).$$

□



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for any finite constant γ.

Lemma 2 (Young's inequality). *If a and b are nonnegative real numbers and p and q are positive real numbers such that $1/p + 1/q = 1$, then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \qquad (1)$$

Special case: $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$

Note. $V = \frac{1}{2}\sum_i^n z_i^2$, $\dot{V} = -\sum_i^n c_i z_i^2$

$$c_{\min} \sum_{i=1}^n x_i^2 \leq c_1 x_1^2 + c_2 x_2^2 + \cdots + c_n x_n^2 \leq c_{\max} \sum_{i=1}^n x_i^2, \qquad (1)$$

where $c_{\min} = \min\{c_1, c_2, c_3, \cdots c_n\}$ *and* $c_{\max} = \max\{c_1, c_2, c_3, \cdots c_n\}$.

Application of Young's inequality. If there exists a term $z_i a_i$ in $\dot{V}_i$ where a_i is a bounded variable or a constant, then we have

$$z_i a_i \leq \frac{1}{2}\varepsilon_i^2 z_i^2 + \frac{1}{2}\frac{a_i^2}{\varepsilon_i^2}$$

with $\varepsilon_i > 0$.

- $\frac{1}{2}\varepsilon_i^2 z_i^2$ can be easily canceled by the virtual control law α_i,
- $\frac{1}{2}\frac{a_i^2}{\varepsilon_i^2}$ is left to the final Lyapunov function V_n as

$$\delta = \sum_{i=1}^n \delta_i = \sum_{i=1}^n \frac{1}{2}\frac{a_i^2}{\varepsilon_i^2} \geq 0.$$

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where $\gamma > 0$ and $\delta > 0$. Define $\rho := \delta/\gamma$,

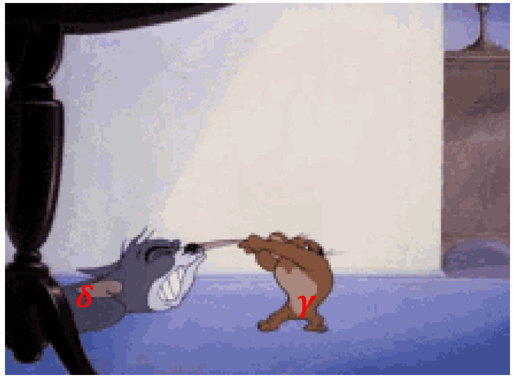
$$0 \leq V(t) \leq \rho + (V(0) - \rho)\exp(-\gamma t). \qquad (2)$$

I am a garbage bin

Note.

$$c_{\min}\sum_{i=1}^{n} x_i^2 \leq c_1 x_1^2 + c_2 x_2^2 + \cdots + c_n x_n^2 \leq c_{\max}\sum_{i=1}^{n} x_i^2, \qquad (1)$$

$c_{\min} = \min\{c_1, c_2, c_3, \cdots c_n\}$ *and* $c_{\max} = \max\{c_1, c_2, c_3, \cdots c_n\}$.



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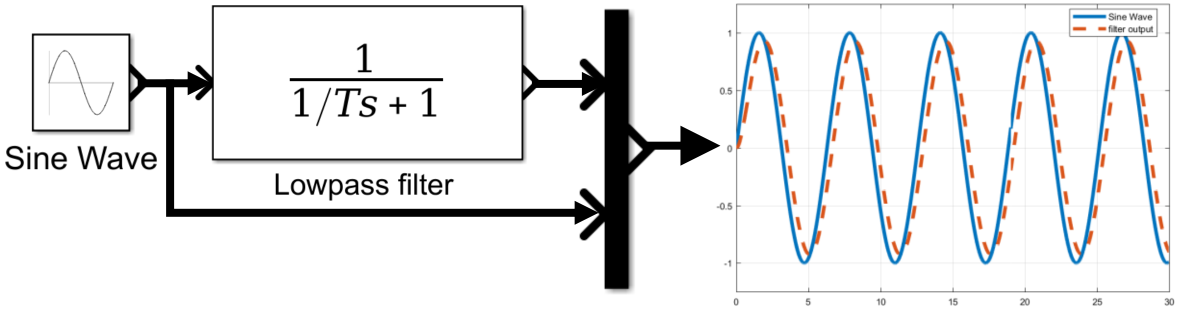
2.1 Dynamic surface control/command filtered backstepping

Introduction

$$\dot{x}_i = f_i(\bar{x}_i) + g_i(\bar{x}_i)x_{i+1} + \phi_i(\bar{x}_i)^\top \theta_i, i \in \mathcal{I}$$
$$\dot{x}_n = f_n(\bar{x}_n) + g_n(\bar{x}_n)u + \phi_n(\bar{x}_n)^\top \theta_n,$$
$$y = x_1,$$

Traditional backstepping technique is subject to "**explosion of complexity**" due to the derivation of multiple sliding surface control scheme.



<u>Repeatedly differentiating</u>

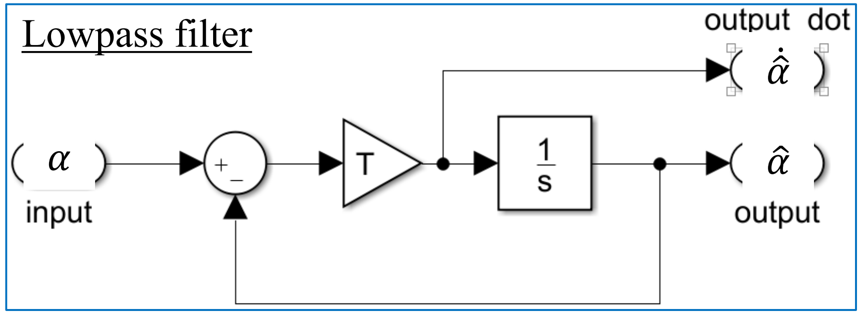
Step 1: $\alpha_1(f_1, \dot{x}_{1d})$

Step 2: $\dot{\alpha}_1 = \frac{\partial \alpha_1}{\partial x_1}\dot{x}_1 + \frac{d\alpha_1}{t}$

$\alpha_2(f_2, \dot{\alpha}_1)$

Step 3: $\dot{\alpha}_2 = \frac{\partial \alpha_2}{\partial x_2}\dot{x}_2 + \frac{d\alpha_2}{dx_1}\dot{x}_1 + \frac{d\alpha_2}{t}$

…



Third-order system is fine. Higher-order systems are nightmares. ☹

Possible solution 1:
Neglecting of high-order terms ↛ Lyapunov stability

Possible solution 2:
Dynamic surface control (DSC) is introduced.

Useful properties of a lowpass filter in control design:

- Output $\hat{\alpha}$ is smooth
- Output converges to input ($\hat{\alpha} \to \alpha$)
- $\dot{\hat{\alpha}}$ is known without derivation
- Larger T $\to$ Smaller error between α and $\hat{\alpha}$
- Nonsmooth $\alpha \to$ smooth $\hat{\alpha}$

2.1 Dynamic surface control/command filtered backstepping

Key process in the deduction

$$\dot{x}_i = f_i(\bar{x}_i) + g_i(\bar{x}_i)x_{i+1}, i \in \mathcal{I}$$
$$\dot{x}_n = f_n(\bar{x}_n) + g_n(\bar{x}_n)u,$$
$$y = x_1,$$

Objects:

- $x_1 \to x_{1d}$ (Traditional backstepping)
- $\alpha_i \to x_{i+1}$ (Traditional backstepping)
- $\hat{\alpha}_i \to \alpha_i$ (New lowpass filter)

(together) $\hat{\alpha}_i \to x_{i+1}$

Define the error states to be:

$$z_1 = x_1 - x_{1d}$$
$$z_i = x_i - \hat{\alpha}_{i-1}$$

Assumptions:

- $x_{1d}(t)$ and its first derivatives $\dot{x}_{1d}$ are known, bounded, and smooth.
- (1) The signs of $g_1, \cdots, g_n$ are assumed to be known and constant; (2) $g_i^{(j)}$ are bounded; (3) $\underline{g}_i < |g_i| < \overline{g}_i$ for all t

Step 1

$$z_1 := x_1 - x_{1d}, V_1 = \frac{1}{2}z_1^2$$
$$\dot{V}_1 = z_1(f_1 + g_1 x_2)$$
$$\alpha_1 = \frac{1}{g_1}(-f_1 - \kappa(z_1))$$
$$T\dot{\hat{\alpha}}_1 = -\hat{\alpha}_1 + \alpha_1, \hat{\alpha}_1(t_0) = \alpha_1(t_0)$$

Step 2

$$z_2 := x_2 - \hat{\alpha}_1$$
$$\dot{z}_2 = \dot{x}_2 - \dot{\hat{\alpha}}_1 = f_2 + g_2 x_3 - \frac{1}{T}(-\hat{\alpha}_1 + \alpha_1)$$
$$\alpha_2 = \frac{1}{g_2}\left(-f_2 - \kappa(z_2) + \frac{1}{T}(-\hat{\alpha}_1 + \alpha_1)\right)$$

…

Swaroop, D., Hedrick, J. K., Yip, P. P., & Gerdes, J. C. (2000). Dynamic surface control for a class of nonlinear systems. *IEEE transactions on automatic control*, *45*(10), 1893-1899.

2.1 Dynamic surface control/command filtered backstepping

Other filters

3. Approximate the virtual control with a filter
- $\tilde{x}_1 := x_1 - x_{1d} \to 0$ as $t \to \infty$
- $\tilde{x}_i := x_{\mathrm{i}} - x_i^c$

where x_i^c is a filtered signal of α_i.

Control objective:
- $x_1 - x_{1d} \to 0$ for $t \to \infty$;
- $|x_i^c - \alpha_{i-1}| \to$ small

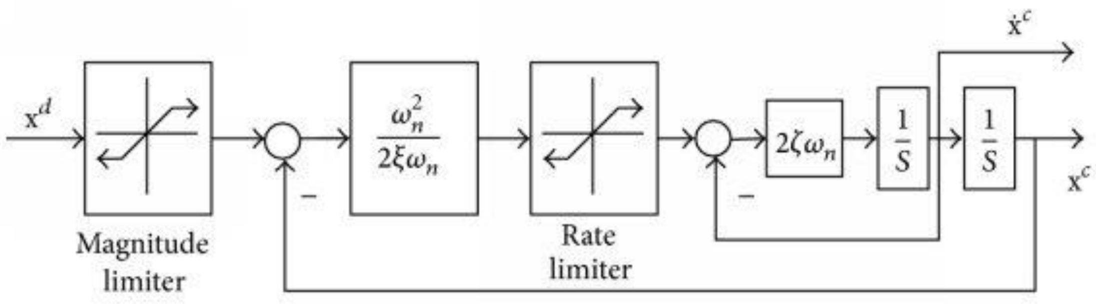
- First-order lowpass $\dot{x}_i^c = -\omega_i(x_i^c - \alpha_{i-1}),$
- Second-order lowpass filter

$$\dot{\varphi}_{i,1} = -\omega_n \varphi_{i,2},$$
$$\dot{\varphi}_{i,2} = -2\zeta\omega_n\varphi_{i,2} - \omega_n(\varphi_{i,1} - \alpha_i),$$

- First-order Levant differentiator with finite-time convergent property

$$\dot{\varphi}_{i,1} = -a_i|\varphi_{i,1} - \alpha_i|^{1/2}, \mathrm{sign}(\varphi_{i,1} - \alpha_i) + \varphi_{i,2},$$
$$\dot{\varphi}_{i,2} = -b_i \mathrm{sgn}(\varphi_{i,2} - \dot{\varphi}_{i,1}),\ i \in \mathcal{I},$$
$$x_{i+1}^c = \varphi_{i,1}$$

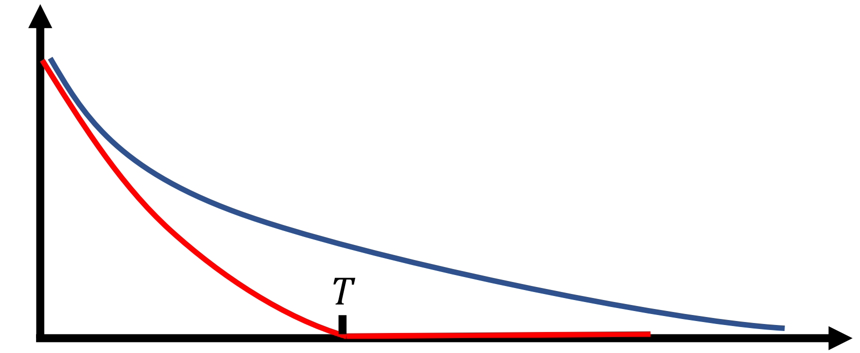
a_i, b_i	Tuned coefficients	ω_n	Natural frequency
ζ	Damping ratio		



2.2 Finite-time control

Finite-time stability

Problem of asymptotical stability: Slow convergence rate near the equilibrium



For a nonlinear system:

$$\dot{x} = f(x), x(0) = x_0 \text{ and } f(x_e) = 0$$

Convergence

	Conditions	Convergence rate
Lyapunov stable	• $\forall \varepsilon > 0, \exists \delta > 0$ such that if $\lvert x(0) - x_e \rvert < \delta$ then we have $\lvert x(t) - x_e \rvert < \varepsilon, \forall t \geq 0$	None solutions starting "close enough" to the equilibrium
Asymptotically stable	• Lyapunov stable • $\exists \delta > 0$ such that if $\lvert x(0) - x_e \rvert < \delta$ then $\lim_{t \to \infty} \lvert x(t) - x_e \rvert = 0$	$t \to \infty, x(t) \to x_e$ Eventually converge to the equilibrium
Exponentially stable	• Asymptotically stable • $\alpha, \beta, \delta > 0$ such that if $\lvert x(0) - x_e \rvert < \delta$ then $\lvert x(t) - x_e \rvert < \alpha \lvert x(0) - x_e \rvert e^{-\beta t}, \forall t \geq 0$	$\lvert x(t) - x_e \rvert < \alpha \lvert x(0) - x_e \rvert e^{-\beta t}$ Converge with an exponential rate
<u>Finite-time stable</u>	• Lyapunov stable • Finite-time convergence $\lim_{t \to T} \lvert x(t) - x_e \rvert = 0$	$t \to T, x(t) \to x_e; x(t) = x_e, \forall t \geq T$ Convergence to the equilibrium in a finite time

2.2 Finite-time control

Stability criteria

For a system $\dot{x} = -c(t)\,\mathrm{sgn}\big(x(t)\big)\ |x(t)|^r$, $r \in (0,1)$, $c > 0$

- Case 1: If $x(0) = 0$, then $x(t) = 0$;
- Case 2: If $x(0) \neq 0$, $\mathrm{sgn}(x)\,\mathrm{sgn}(x) = 1$,

$$\frac{dx}{dt} = -c(t)\,\mathrm{sgn}\big(x(t)\big)\ |x(t)|^r \Rightarrow \frac{dx}{\mathrm{sgn}\big(x(t)\big)\ |x(t)|^r} = -c(t)dt \qquad (1)$$

If $r \in (0,1)$,

$$\frac{\mathrm{d}}{dx}|x(t)|^{1-r} = \frac{\mathrm{d}}{dx}\Big(\mathrm{sgn}\big(x(t)\big)\,x(t)\Big)^{1-r}$$

$$= (1-r)\Big(\mathrm{sgn}\big(x(t)\big)\,x(t)\Big)^{-r}\left(\frac{\mathrm{d\,sgn}\big(x(t)\big)}{dx}x(t) + \mathrm{sgn}\big(x(t)\big)\frac{\mathrm{d}x(t)}{dx}\right)$$

$$= \mathrm{sgn}\big(x(t)\big)\,(1-r)|x(t)|^{-r}$$

Integrating both sides of (1) :

$$-\int_0^t c(t)\,d\tau = \frac{|x|^{1-r}}{1-r}\bigg|_0^t = \frac{|x(t)|^{1-r}}{1-r} - \frac{|x(0)|^{1-r}}{1-r}$$

$$|x(t)|^{1-r} = |x(0)|^{1-r} - (1-r)\int_0^t c(t)\,d\tau$$

$$\mathrm{sgn}\big(x(t)\big)\,x(t) = \left(|x(0)|^{1-r} - (1-r)\int_0^t c(t)\,d\tau\right)^{\frac{1}{1-r}}$$

$$x(t) = \mathrm{sgn}\big(x(t)\big)\left(|x(0)|^{1-r} - (1-r)\int_0^t c(t)\,d\tau\right)^{\frac{1}{1-r}}$$

Therefore, $\exists\, T\ s.t.\ x(T) = 0$ if $c(t) > 0$. If c is a constant, $T = \frac{|x(0)|^{1-r}}{c(1-r)}$

Replace $x(t)$ by a LFC $V(t)$ and replace $c(t)$ by γ

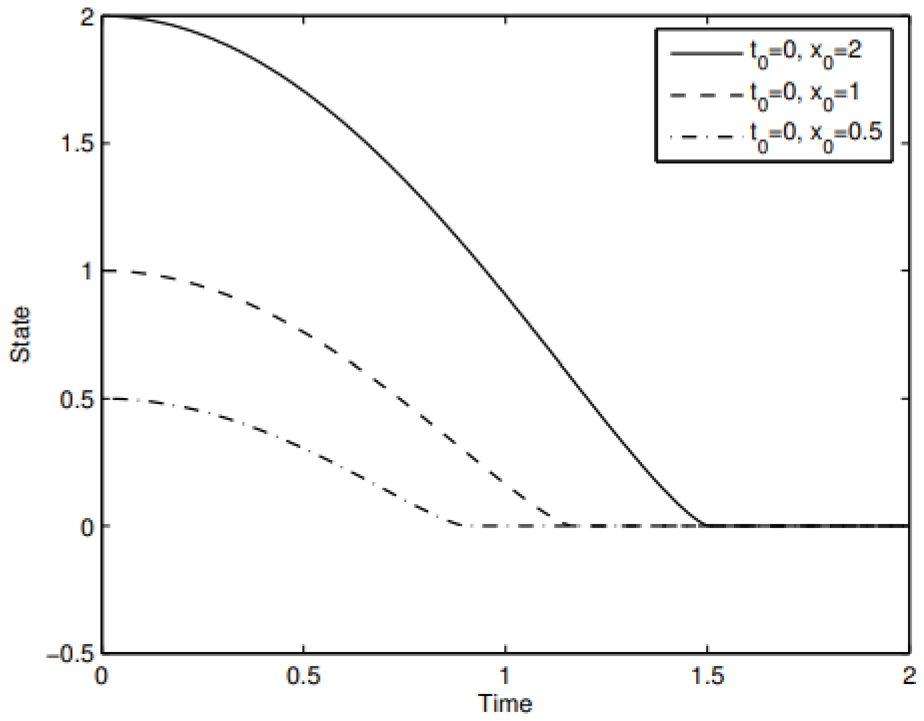
$$\dot{V} = -\gamma\,\mathrm{sgn}\big(V(t)\big)\ |V(t)|^r$$

Since $V(t) \geq 0$, $\mathrm{sgn}\big(V(t)\big) \geq 0$, a finite-time Lyapunov stability criteria is received, i.e.,

$$\dot{V} = -\gamma V^r.$$

Then,

$$T = \frac{V(0)^{1-r}}{\gamma(1-r)}$$



2.2 Finite-time control

Stability criteria and key inequalities

Lyapunov-like stability criteria

The origin is a finite-time-stable equilibrium if there exists a continuous positive definite function $V(x)$, real numbers $\gamma > 0$, and $r \in (0, 1)$, such that,

$$\dot{V}(x) \leq -\gamma V^{r}(x), \forall x \in \mathbb{N}\backslash\{0\},$$

where $\gamma > 0$ and $\mathbb{N}$ is an open neighborhood of the origin. The settling-time function is a function of the initial value of the LF $T \leq \frac{1}{\gamma(1-r)} V(x_0)^{1-r}, x \in \mathbb{N}$.

Extended forms with better convergence when $x \gg 1$:

- $\dot{V}(x) \leq -\gamma_1 V(x) - \gamma_2 V^r(x)$
  $T = \frac{1}{\gamma_1(1-r)} \ln \frac{\gamma_1 V^{1-r}(x_0)+\gamma_2}{\gamma_2}$
- $\dot{V}(x) \leq -\gamma_1 V^{r'}(x) - \gamma_2 V^r(x)$
  $T = \frac{1}{\gamma_2(1-r)} + \frac{V^{1-r'}(x_0)-1}{\gamma_1(1-r')}$

where $\gamma_1 > 0$, $\gamma_2 > 0$ and $r' > 1$

Finite-time stability in probability $\mathcal{L}V(x) \leq -\gamma V^r(x)$

Lemmas using for parameter separation

Lemma[FT1] *For any $x_i \in \mathbb{R}$, $i = 1, \cdots, n$ and a real number $a \in (0, 1]$, the following inequality holds*

$$\left(\sum_{i=1}^{n}|x_i|\right)^p \leq \sum_{i=1}^{n}|x_i|^p \leq n^{1-p}\left(\sum_{i=1}^{n}|x_i|\right)^p.$$

Lemma[FT3] *For $x \in \mathbb{R}$, $y \in \mathbb{R}$, and p is an integer, the following inequality holds*

$$|x+y|^p \leq 2^{p-1}|x^p + y^p|,$$

$$(|x|+|y|)^{1/p} \leq |x|^{1/p} + |y|^{1/p} \leq 2^{p-1/p}(|x|+|y|)^{1/p}$$

If $p \geq 1$ is an odd integer, then

$$|x-y|^p \leq 2^{p-1}|x^p - y^p|.$$

Lemma[FT3] *Let a and b be positive real number and $\gamma(x, y) > 0$ be a real-value function. Then,*

$$|x|^a|y|^b \leq \frac{a}{a+b}\gamma(x,y)|x|^{a+b} + \frac{b}{a+b}\gamma(x,y)|y|^{a+b}.$$

If $x \geq 0$, $y \geq 0$, and $\pi \geq 0$ are continuous, then for any constant $c > 0$,

$$|x|^a|y|^b \leq c|x|^{a+b} + \frac{b}{a+b}\left[\frac{a}{c(a+b)}\right]^{a/b}|y|^{a+b}\pi^{(a+b)/b}.$$

Lemma[FT4] *Let a and b be positive real number and $\gamma(x, y) > 0$ be a real-value function. Then,*

$$|x|^a|y|^b \leq \frac{a}{a+b}\gamma(x,y)|x|^{a+b} + \frac{b}{a+b}\gamma(x,y)|y|^{a+b}$$

2.2 Finite-time control

Deduction keypoints

Commonly-used LFC:

- [FT1] $V_{i,FT} = \int_{\alpha_i}^{x_i} \left(s^{1/q_k} - \alpha_i^{1/q_i}\right) ds$
- [FT2] $V_{i,FT} = \int_{\alpha_i}^{x_i} \left(s^{\beta_{i-1}} - \alpha_{i-1}^{\beta_{i-1}}\right) ds$
- [FT3] $V_{i,FT} = \int_{\alpha_i}^{x_i} \left(s^{1/q_k} - \alpha_i^{1/q_i}\right)^{2-q_k} ds$ (order-r_i)
- [FT4] $V_{i,FT} = \int_{\alpha_i}^{x_i} \left(s^{1/q_k} - \alpha_i^{1/q_i}\right)^{2-q_k-\tau} ds$ (order-1)
  where τ is a ratio of two numbers.

Assumptions:

Parameter separation

- $|f_i| \leq (\sum_{j=1}^{i} |x_j|)\rho_i(\bar{x}_i)$
- $|f_i| \leq \frac{1}{2}|x_{i+1}|^{r_i} + \sum_{j=1}^{i} |x_j|\rho_i(\bar{x}_i))\sigma$
- $|f_i| \leq \sigma\rho_i(\bar{x}_i))$
- $|f_i| = \varphi(t) \sum_{j=1}^{i} |x_j|^{m_{ij}} + \sigma \sum_{j=1}^{i} |x_j|^{n_{ij}}$ (time-varying system)

where $\rho_i(\bar{x}_i)$ are smooth known C^1 positive functions and $\sigma \geq 1$ is an uncertain constant.

- $r_1 > \cdots > r_n$ since the higher-dimension dynamics should react faster than the lower dimension.
- Recursive design approach → inductive design approach

Debate of practical finite-time stability

Similar to Lemma 1, practical finite-time stability is proposed with additional term δ, i.e.,

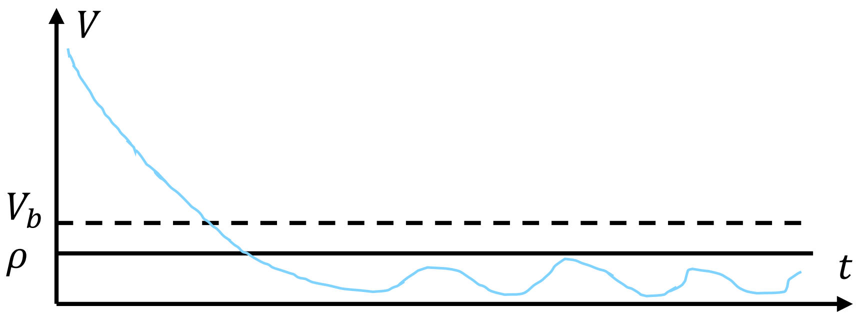
$$\dot{V} \leq \gamma V^{\alpha} + \delta.$$

The tracking error converges to a disk region and remains in the region in finite time.

However, $\dot{V} \leq \gamma V + \delta$ can also ensure the convergence to a disk region V_b in finite time. If we set the boundary value to be $V(T) = V_b \geq \rho$, then the settle time to V_b is

$$0 \leq V(t) \leq \rho + (V_0 - \rho)\exp(-\lambda t)$$

$$T = -\frac{1}{\lambda}\ln\left(\frac{V_b - \rho}{V_0 - \rho}\right) = -\frac{1}{\lambda}\ln\left(\frac{\lambda V_b - \delta}{\lambda V_0 - \delta}\right)$$



2.3 Approximation-based backstepping

Neural network and fuzzy logic system

$$\dot{x}_i = {\color{red}f_i(\bar{x}_i)} + g_i(\bar{x}_i)x_{i+1}, i \in \mathcal{I}$$

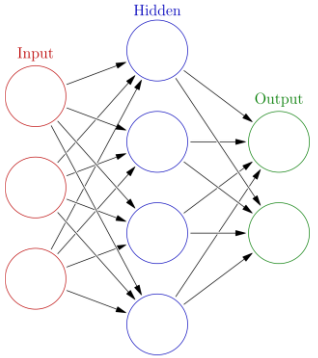
$$\dot{x}_n = {\color{red}f_n(\bar{x}_n)} + g_n(\bar{x}_n)u,$$

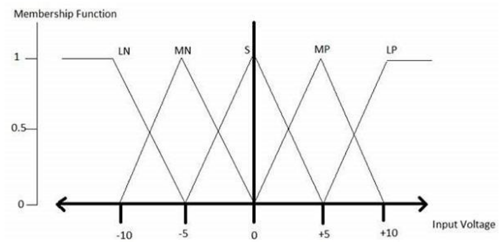
$$y = x_1,$$

Control object: $x_1 - x_{1d} \to 0$ for $t \to \infty$;
Additional problem: f_i is unknown
Idea: (1) Approximate all uncertainty with learning,
(2) Cancel the estimated values in α_i and u

Universal approximation property: Any smooth function in a compact set can be approximated by an NN with arbitrary small error by sufficiently large number of nodes

Neural network (NN) | Fuzzy logic system (FLS)





- Their orientations are different, but their mathematical deductions are very similar.
- FLS and NN are ways to find ψ_i. There is no need to design the candidate functions in ψ_i.

- Two-layer radial basis function NN (RNFNN)

$$f(x) = W^\top S(x) + \varepsilon$$

- Multilayer neural networks (MNN) three-layer Wavelet NN (WNN)

$$f(z) = W^\top S(D^\top z + T) + \varepsilon \xRightarrow{U^\top = [D^\top, T],\ \bar{z} = [z^\top, 1]^\top} W^\top S(U^\top \bar{z}) + \varepsilon$$

First-to-second layer weight vector	$W = [w_1, \cdots, w_l]^\top \in \mathbb{R}^l$
Second-to-third layer weight vector	$V = [v_1, \cdots, v_l]^\top \in \mathbb{R}^{q \times l}$
Corresponding reconstruction error	ε
$S(z) = [s_{1(z)}, \cdots, s_{l(z)}]^\top$	$T = [t_1, \cdots, t_l]^\top \in \mathbb{R}^l$

$f(x) = \theta^\top \varphi(x) + \varepsilon$	$\varepsilon \le \bar{\varepsilon}, \bar{\varepsilon}$ is a
$\theta = [\bar{y}_1, \cdots, \bar{y}_N]^\top$	$\bar{y}_l = s_i = \max_{y \in \mathbb{R}} \mu_{G^l}(y)$
Fuzzy-membership function	$\mu_{G^l}(y) = \exp\left(\frac{x_i - a_i^l}{b_i^l}\right)$
	$\varphi^\top = [\varphi_1, \cdots, \varphi_N]$
Fuzzy basic function	$\varphi_l = \frac{\prod_{i=1}^N \mu_i^l(x_i)}{\sum_{l=1}^N \prod_{i=1}^N \mu_i^l(x_i)}$
$0 \le \varphi^\top \varphi \le 0$	$z_i \theta^\top \varphi \le \frac{z_i^2}{4\lambda} \lvert\theta\rvert^2 + \lambda$

2.3 Approximation-based backstepping

Using neural adaptive backstepping as an example

$$\dot{x}_i = f_i(\bar{x}_i) + g_i(\bar{x}_i)x_{i+1}, i \in \mathcal{I}$$
$$\dot{x}_n = f_n(\bar{x}_n) + g_n(\bar{x}_n)u,$$
$$y = x_1,$$

Control object: $x_1 - x_{1d} \to 0$ for $t \to \infty$;
Additional problem: f_i is unknown
Idea: (1) Approximate all uncertainty with learning,
(2) Cancel the estimated values in α_i and u

Assumptions: W is bounded with known W_m, i.e., $\|W\|_F \le W_m$.

+ No need to design explicit basis functions
÷ Lack of capacity to extract the underlying structures of the nonlinear functions.
÷ Long learning time resulting from the significant number of NN nodes and adaption parameters to receive sufficient approximation accuracy.
÷ Explosion of states. Most examples in the case studies are second-order systems. (A possible solution: $|W|$ can be used instead of W).
÷ Local stability since NN approximation is only valid in specific compact sets.

The deduction is similar to the typical adaptive backstepping

Assume: $f(x) = \widehat{W}^\top S(x) + \varepsilon$

Deduction remarks:

- Define the error vector of weights: $\widetilde{W}_i = W_i - \widehat{W}_i$
- LFC: $V_i = V_{i-1} + z_i^2 + \frac{1}{2}\widetilde{W}_i^\top \Gamma_i^{-1}\widetilde{W}_i$
- $f_i = W_i^\top S(\bar{x}_i) = \widehat{W}_i^\top S(\bar{x}_i) + \widetilde{W}_i^\top S(\bar{x}_i)$
- $\dot{V}_i = \kappa(\bar{z}_{i-1}) + z_i\left(\ldots + \widehat{W}_i^\top S(\bar{x}_i) + \widetilde{W}_i^\top S(\bar{x}_i) + g_i x_{i+1}\right) + \frac{1}{2}\widetilde{W}_i^\top \Gamma_i^{-1}\dot{\widetilde{W}}_i$
- Virtual control: $\alpha_i = \frac{1}{g_i}\left(-\cdots - \widehat{W}_i^\top S(\bar{x}_i) - k_i z_i\right)$
- Substitute α_i into $\dot{V}_i$: $\dot{V}_i = \kappa(\bar{z}_i) + z_i\widetilde{W}_i^\top S(\bar{x}_i) + \widetilde{W}_i^\top \Gamma_i^{-1}\dot{\widetilde{W}}_i + g_i z_i z_{i+1}$
- Adaptive update law with a σ-modification:
$$\dot{\widetilde{W}}_i = -\Gamma_i S(\bar{x}_i) z_i - \Gamma_i \sigma \widehat{W}$$
- Substitute into $\dot{V}_i$ and apply Young's inequality:
$$\widetilde{W}^\top \widehat{W} = \widetilde{W}^\top W - \widetilde{W}^\top \widetilde{W} \le \frac{1}{2}\left(-\widetilde{W}^\top \widetilde{W} + W^\top W\right)$$
- $\dot{V}_n \le -\gamma V_n + \delta$ (But not $\dot{V} \le 0$ since there may exist other nonlinearies.)
- According to the assumption $W^\top W$ is bounded.

Tip: The theory is simple but the relevant journals has a much higher impact factor.

2.4 Nussbaum function

Introduction

$$\dot{x}_i = f_i(\bar{x}_i) + g_i x_{i+1} + \phi_i(\bar{x}_i)^\top \theta, i \in \mathcal{I}$$
$$\dot{x}_n = f_n(\bar{x}_n) + g_n u + \phi_n(\bar{x}_n)^\top \theta,$$
$$y = x_1,$$

Control object: $x_1 - x_{1d} \to 0$ for $t \to \infty$;
Additional problem: Unknown control coefficient g_i

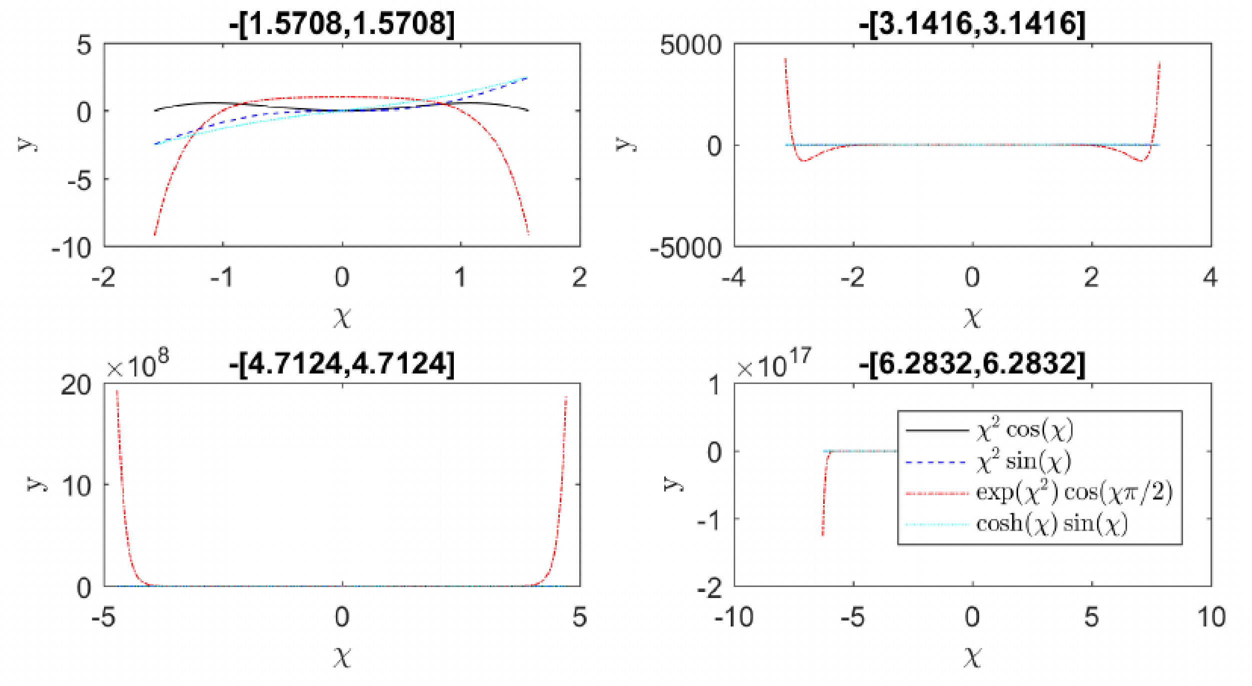
Definition (Nussbaum-gain function)

$$\lim_{s\to\infty} \sup \frac{1}{s} \int_0^s \mathcal{N}(\chi) d\chi = +\infty$$
$$\lim_{s\to\infty} \inf \frac{1}{s} \int_0^s \mathcal{N}(\chi) d\chi = -\infty$$
$$\mathcal{M}(\chi) := \int_0^\chi \mathcal{N}(\tau) d\tau$$

1. Amplitude-elongation Nussbaum-type functions are commonly adopted which are the products of a class K_∞ function and a trigonometric function, for example,

- $\mathcal{N}(\chi) = \chi^2 \cos(\chi)$,
- $\mathcal{N}(\chi) = \chi^2 \sin(\chi)$,
- $\mathcal{N}(\chi) = \exp(\chi^2) \cos(\chi\pi/2)$
- $\mathcal{N}(\chi) = \frac{1}{2} e^{-\sigma\chi} \cos\chi - \frac{1}{2} e^{\sigma\chi} \cos\chi$
- $\mathcal{N}(\chi) = \cosh(\lambda\xi) \sin(\xi)$
- $\mathcal{N}(\chi) = \exp(\xi^2/2)(\xi^2 + 2) \sin(\chi)$.

2. Time-elongation



Key lemma:
Let $V(t)$ and $\chi(t)$, $i = 1,2,\cdots,n$, be smooth functions defined on $[0, t_f)$ with $V(t) \geq 0$ and $\chi_i(0)$=0. If the following inequality holds

$$V(t) \leq c_0 + e^{-c_1 t} \int_0^t g_1 \mathcal{N}(\chi)\dot{\chi} e^{c_1\tau} + \dot{\chi} e^{c_1\tau} d\tau, \quad (1)$$

where $c_1 > 0$. Then $V(t)$, $\chi(t)$, and $\int_0^t g(\tau)\mathcal{N}(\chi) d\tau$ must be bounded on $[0, t_f)$.

Eq. (1) is the results of the following form

$$\dot{V} \leq -c_1 V + g_1 \mathcal{N}(\chi)\dot{\chi} + \dot{\chi} + \delta. \quad (2)$$

Times $e^{c_1 t}$ to both sides and $\frac{d}{dt}(V e^{c_1 t}) = \dot{V} e^{c_1 t} + c_1 V e^{c_1 t}$

$$c_0 = \frac{c_0}{c_1} + e^{-c_1 t} V(0) + e^{-c_1 t} \int_0^t \delta e^{c_1 \tau} d\tau$$

2.4 Nussbaum function

Example

$$\dot{x}_i = f_i(\bar{x}_i) + g_i x_{i+1} + \phi_i(\bar{x}_i)^\top \theta, i \in \mathcal{I}$$
$$\dot{x}_n = f_n(\bar{x}_n) + g_n u + \phi_n(\bar{x}_n)^\top \theta,$$
$$y = x_1,$$

Control object: $x_1 - x_{1d} \to 0$ for $t \to \infty$;
Additional problem: Unknown control coefficient g_i

Using the LFC as $V_1 = V_{1,QF} + \frac{1}{2}\tilde{\theta}_1^\top \Gamma_1^{-1}\tilde{\theta}_1$. Then its time derivative is given by

$$\begin{aligned}\dot{V}_1 =& z_1[f_1 + g_1(\alpha_1 + z_2) + \phi_1^\top \theta - \dot{x}_{1d}] - \tilde{\theta}_1^\top \Gamma_1^{-1}\dot{\hat{\theta}}_1 \\ =& g_1 z_1 z_2 + g_1 z_1 \alpha_1 + z_1(f_1 + \hat{\theta}_1^\top \phi_1 - \dot{x}_{1d}) \\ & - \tilde{\theta}_1^\top(\Gamma_1^{-1}\dot{\hat{\theta}}_1 - z_1\phi_1) \\ \le & z_1(f_1 + \hat{\theta}_1^\top \phi_1 - \dot{x}_{1d} + \frac{1}{4}z_1) + g_1 z_1 \alpha_1 + g_1^2 z_2^2 \\ & - \tilde{\theta}_1^\top(\Gamma_1^{-1}\dot{\hat{\theta}}_1 - z_1\phi_1)\end{aligned}$$

The desired form: $\dot{V} \le -c_1 V + g_1 \mathcal{N}(\chi)\dot{\chi} + \dot{\chi} + \delta$

Understanding: $\mathcal{N}(\chi_1)$ amplifies the control input if η_1 is not enough. The core idea is robustness-based.

Let

$$\eta_1 := c_1 z_1 + f_1 + \hat{\theta}_1^\top \phi_1 - \dot{x}_{1d} + \frac{1}{4}z_1$$

$$\dot{\chi}_1 := z_1 \eta_1$$

Similar to virtual control in classic design without $-\frac{1}{g_i}$

Virtual control law

$$\alpha_1 = \mathcal{N}(\chi_1)\eta_1.$$

Adaptive update law

$$\dot{\hat{\theta}}_1 = \gamma_1 \mathrm{Proj}(z_1\phi_i, \hat{\theta}_1).$$

then the stability can be prove with Lemma

$$\dot{V}_1 \le -c_1 z_1^2 + \dot{\chi}_1 + g_1 \mathcal{N}(\chi_1)\dot{\chi}_1 + g_1^2 z_2^2$$

Integrating over $[0, t]$ and apply the Lemma of Nussbaum function.

Step i: $V_i = V_{i,QF} + \frac{1}{2}\tilde{\theta}_i \Gamma_i^{-1}\tilde{\theta}_i$ (V_{i-1} is not included.)

$$\dot{V}_i = -c_i V_i + \dot{\chi}_i + g_i \mathcal{N}(\chi_i) + g_i^2 z_{i+1}^2 \quad \text{(i)}$$

Step n: $V_i = V_{i,QF} + \frac{1}{2}\tilde{\theta}_n \Gamma_n^{-1}\tilde{\theta}_n$ (V_{n-1} is not included.)

$$\dot{V}_i = -c_n V_n + \dot{\chi}_n + g_n \mathcal{N}(\chi_n) \quad \text{(n)}$$

Proof: * Start from step n. Times $e^{c_n t}$ to both sides of (n) and integrate. Then we know V_n is bounded ⇒ z_n is bounded.
*Step n-1: Times $e^{c_{n-1}t}$ to both sides of (n-1) and integrate. $e^{-c_1 t} g_{n-1}^2 z_n^2 \int_0^t e^{c_1\tau} d\tau$ is bounded. Then V_{n-1} is bounded ⇒ z_n is bounded.
*Step n-2; ….; Step 1, V_1 is bounded ⇒ z_1 is bounded.

2.5 Barrier Lyapunov function (BLF)

Introduction

$$\dot{x}_i = f_i(\bar{x}_i) + g_i(\bar{x}_i)x_{i+1}, i \in \mathcal{I}$$
$$\dot{x}_n = f_n(\bar{x}_n) + g_n(\bar{x}_n)u,$$
$$y = x_1,$$

Control object: $x_1 - x_{1d} \to 0$ for $t \to \infty$;
Additional problem: Output constraints $-k_{a1} < z_1 < k_{b1}$
State constraints $-k_{ai} < z_i < k_{bi}$

Why quadratic Lyapunov function cannot guarantee state constraint?

$$V(x) = \sum_{i=1}^{n} \frac{1}{2} z_i^2$$

Then, $V(t) = \sum_{i=1}^{n} \frac{1}{2} z_i^2 \leq V(t_0)$.
We can conclude that $\sqrt{z_1^2 + z_2^2 + \cdots + z_n^2} \leq \sqrt{2V(t_0)}$.
The upper limit of $z_1 \leq \sqrt{2V(t_0)}$.
Recall Lemma 1, $z_1(t) \leq \sqrt{2V(t)}$.
But the value of z_1 is only limited by the initial value of the Lyapunov function. Manually setting the constraints is impossible.

Lyapunov function:
- $V(z) = 0$ iff $z = 0$
- $V(z) > 0$ iff $z \neq 0$
- $\dot{V}(z) \leq 0, \forall z \neq 0$

Lemma 1. *A LFC $V(x)$ is bounded if the initial condition $V(0)$ is bounded, $V(x)$ is positive definite and continuous and if a Lyapunov-like inequality holds, i.e.,*

$$\dot{V}(x) \leq -\gamma V(x) + \delta, \qquad (1)$$

where $\gamma > 0$ and $\delta > 0$. Define $\rho := \delta/\gamma$,

$$0 \leq V(t) \leq \rho + (V(0) - \rho)\exp(-\gamma t). \qquad (2)$$

And it implies that

$$V(t) \leq e^{-\gamma t}V(0) + \int_0^t e^{-\gamma(t-\tau)}\rho(\tau)d\tau, \ \forall t \geq 0, \qquad (3)$$

for any finite constant γ.

2.5 Barrier Lyapunov function (BLF)

Key lemmas

Lemma[1] *For any positive constants k_{a_1}, k_{b_1}, let $\mathcal{Z}_1 := \{z_1 \in \mathbb{R} : -k_{a_1} < z_1 < k_{b_1}\} \subset \mathbb{R}$ and $\mathcal{N} := \mathbb{R}^l \times \mathcal{Z}_1 \subset \mathbb{R}^{l+1}$ be open sets. Consider the system*

$$\dot{\eta} = h(t, \eta) \tag{3}$$

where $\eta := [w, z_1]^T \in \mathcal{N}$, and $h : \mathbb{R}_+ \times \mathcal{N} \to \mathbb{R}^{l+1}$ is piecewise continuous in t and locally Lipschitz in z, uniformly in t, on $\mathbb{R}_+ \times \mathcal{N}$. Suppose that there exist functions $U : \mathbb{R}^l \to \mathbb{R}_+$ and $V_1 : \mathcal{Z}_1 \to \mathbb{R}_+$, continuously differentiable and positive definite in their respective domains, such that

$$V_1(z_1) \to \infty \quad \text{as } z_1 \to -k_{a_1} \quad \text{or} \quad z_1 \to k_{b_1} \tag{4}$$

$$\gamma_1(\|w\|) \le U(w) \le \gamma_2(\|w\|) \tag{5}$$

where γ_1 and γ_2 are class K_∞ functions. Let $V(\eta) := V_1(z_1) + U(w)$, and $z_1(0)$ belong to the set $z_1 \in (-k_{a_1}, k_{b_1})$. If the inequality holds:

$$\dot{V} = \frac{\partial V}{\partial \eta} h \le 0 \tag{6}$$

then $z_1(t)$ remains in the open set $z_1 \in (-k_{a_1}, k_{b_1})$ $\forall t \in [0, \infty)$.

Lemma[2] For any positive constant k_{b_1}, let $\mathcal{Z}_1 := \{z_1 \in \mathbb{R} : |z_1| < k_{b_1}\} \subset \mathbb{R}$ and $\mathcal{N} := \mathbb{R}^l \times \mathcal{Z}_1 \subset \mathbb{R}^{l+1}$ be open sets. Consider the system

$$\dot{\eta} = h(t, \eta) \tag{6}$$

where $\eta := [w, z_1]^T \in \mathcal{N}$ is the state, and the function $h : \mathbb{R}_+ \times \mathcal{N} \to \mathbb{R}^{l+1}$ is piecewise continuous in t and locally Lipschitz in z_1, uniformly in t, on $\mathbb{R}_+ \times \mathcal{N}$. Suppose that there exist continuously differentiable and positive definite functions $U : \mathbb{R}^l \to \mathbb{R}_+$ and $V_1 : \mathcal{Z}_1 \to \mathcal{R}_+$, $i = 1, \ldots, n$, such that

$$V_1(z_1) \to \infty \quad \text{as} \quad |z_1| \to k_{b_1} \tag{7}$$

$$\gamma_1(\|w\|) \le U(w) \le \gamma_2(\|w\|) \tag{8}$$

with γ_1 and γ_2 as class K_∞ functions. Let $V(\eta) := V_1(z_1) + U(w)$, and $z_1(0) \in \mathcal{Z}_1$. If the inequality holds

$$\dot{V} = \frac{\partial V}{\partial \eta} h \le -\mu V + \lambda \tag{9}$$

in the set $\eta \in \mathcal{N}$ and μ, λ are positive constants, then w remains bounded and $z_1(t) \in \mathcal{Z}_1$, $\forall t \in [0, \infty)$.

Human words:

Choose a Lyapunov function $V = V_1(z_1) + \sum_{i=2}^{n} \frac{1}{2} z_i^2$, and V_1 satisfies:

- $V_1(z_1) \to \infty$ when $z_1 \to -k_{a1}$ or $z_1 \to k_{b1}$
- $-k_{a1} < z_1(t_0) < k_{b1}$

Since $V(t) \le V(t_0) < \infty$ and $\sum_{i=2}^{n} \frac{1}{2} z_i^2 \ge 0$, $V_1 \le V(t_0) < \infty$.

Therefore, z_1 must stay within $(-k_{a1}, k_{b1})$

[1] Tee, K. P., Ge, S. S., & Tay, E. H. (2009). Barrier Lyapunov functions for the control of output-constrained nonlinear systems. *Automatica*, *45*(4), 918-927

[2] Ren, B., Ge, S. S., Tee, K. P., & Lee, T. H. (2010). Adaptive neural control for output feedback nonlinear systems using a barrier Lyapunov function. *IEEE Transactions on Neural Networks*, *21*(8), 1339-1345.

2.5 Barrier Lyapunov function (BLF)

Definition

$$\dot{x}_i = f_i(\bar{x}_i) + g_i(\bar{x}_i)x_{i+1}, i \in \mathcal{I}$$
$$\dot{x}_n = f_n(\bar{x}_n) + g_n(\bar{x}_n)u,$$
$$y = x_1,$$

Control object: $x_1 - x_{1d} \to 0$ for $t \to \infty$;

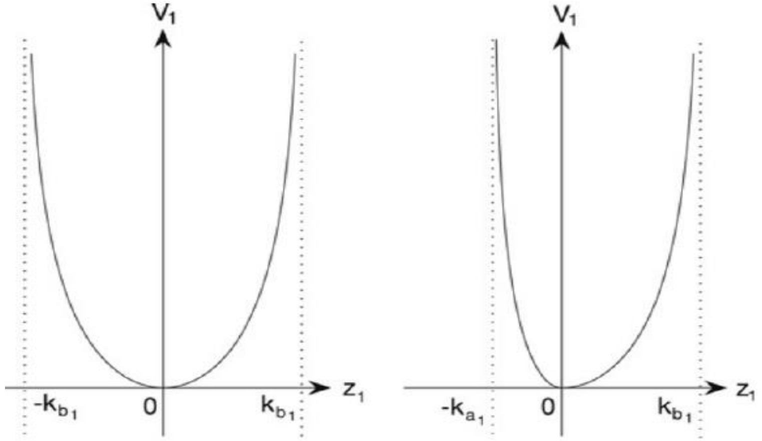
Additional problem: Output constraints $-k_{a1} < z_1 < k_{b1}$

State constraints $-k_{ai} < z_i < k_{bi}$

Idea: $V \to \infty$, when x is close to the barriers.

Assumptions:

- When $x_1(t) \in \mathbb{D}_{x1}$, $|g_i(\bar{x}_i)| > g_0 > 0$
- $\exists A_0 > 0, \underline{x}_{1d} > 0, \bar{x}_{1d} > 0$, and $\bar{x}_{1d} > 0$, $i = 2, \cdots, n$ satisfying $\max\{\underline{x}_{1d}, \bar{x}_{1d}\} \le A_0 \le k_{c1}$, $\underline{x}_{1d} \le x_{1d} \le \bar{x}_{1d}$, and $x_{1d}^{(i)} < \bar{x}_{id}, \forall k_{c1} > 0$ and $t \ge 0$.



- [BLF1] Symmetric barriers ($k_{ai} = k_{bi}$)
  - $V_{i,BLF}(z_i) = \frac{1}{2}\log\frac{k_{bi}^2}{k_{bi}^2 - z_i^2} = \frac{1}{2}\log\frac{1}{1-\xi_b^2}$,
  - $\dot{V}_{i,BLF}(z_i) = \frac{z_i}{k_{bi}^2 - z_i^2}\dot{z}_i$,
- [BLF2] Asymmetric barriers, ($k_{ai} \neq k_{bi}$)
  - $V_{i,BLF}(z_i) = \frac{q}{p}\log\frac{k_{bi}^p}{k_{bi}^p - z_i^p} + \frac{1-q}{p}\log\frac{k_a^p}{k_{ai}^p - z_i^p}$,
  - $\dot{V}_{i,BLF}(z_i) = \frac{q z_i^{p-1}}{k_{bi}^p - z_i^p}\dot{z}_i + \frac{(1-q)z_i^{p-1}}{k_{ai}^p - z_i^p}\dot{z}_i$,
- [BLF3] Time-varying constrain situation ($\dot{k}_{ai} \neq 0$ and $\dot{k}_{bi} \neq 0$)
  - $V_{i,BLF}(z_i) = \frac{q}{p}\log\frac{1}{1-\xi_b^p} + \frac{1-q}{p}\log\frac{1}{1-\xi_a^p}$,
  - $\dot{V}_{i,BLF}(z_i) = \frac{q\xi_b^{2p-1}}{k_{bi}(1-\xi_b^2 p)}(\dot{z}_i - \frac{z_i}{k_{bi}}\dot{k}_{bi}) + \frac{(1-q)\xi_a^{2p-1}}{k_{ai}(1-\xi_a^2 p)}(\dot{z}_i - \frac{z_i}{k_{ai}}\dot{k}_{ai})$,

  $\xi = q\xi_b + (1-q)\xi_a$ → $V_{i,BLF}(z_i) = \frac{1}{2p}\log\frac{1}{1-\xi^{2p}}$
- Other types
  - $V_{i,BLF} = \frac{k_{bi}^2}{\pi}\tan(\frac{\pi z_i^2}{2k_{bi}^2})$
  - $V_{i,BLF} = \cot\frac{\pi}{2}(1 - (\frac{z_i}{k_{bi}})^2)$

$p \ge n$ is an even integer

$\xi_a = \frac{z_i}{k_{ai}}, \xi_b = \frac{z_i}{k_{bi}}$

$q(z_i) = \begin{cases} 1, \text{if } z_i > 0 \\ 0, \text{if } z_i \le 0 \end{cases}$

2.5 Barrier Lyapunov function

Example

$$\dot{x}_i = f_i(\bar{x}_i) + g_i(\bar{x}_i)x_{i+1}, i \in \mathcal{I}$$
$$\dot{x}_n = f_n(\bar{x}_n) + g_n(\bar{x}_n)u,$$
$$y = x_1,$$

Control object: $x_1 - x_{1d} \to 0$ for $t \to \infty$;
Additional problem: Output constraints $-k_{b1} < z_1 < k_{b1}$
Idea: $V \to \infty$, when x is close to the barriers.

Step 1: The LFC with constrained z_1 is given by

$$V_1 = V_{1,BLF} = \frac{1}{2}\log\frac{k_{b1}^2}{k_{b1}^2 - z_1^2}, \quad (1a)$$

The derivative of V_1

$$\dot{V}_1 = \frac{z_i}{k_{bi}^2 - z_i^2}(f_1 + g_1 x_2 - \dot{x}_{1d}),$$

Then the virtual control is

$$\alpha_1 = \frac{1}{g_1}[-f_1 + \dot{x}_{1d} - c_1(k_{b1}^2 - z_1^2)z_1]. \quad (2)$$

Then, substituting α_1 into $\dot{V}_1$ yields

$$\dot{V}_1 = -c_1 z_1^2 + \frac{g_1 z_1 z_2}{k_{b1}^2 - z_1^2}. \qquad \dot{V} \le 0 \quad (3)$$

Key inequality:
For all $|\xi| < 1$ and any positive integer p, $\log\frac{1}{1-\xi^{2p}} < \frac{\xi^{2p}}{1-\xi^{2p}}$.
If $\xi = \frac{z_i}{k_{bi}}$, $\log\frac{k_{bi}^{2p}}{k_{bi}^{2p}-z_i^{2p}} < \frac{z_i^{2p}}{k_{bi}^{2p}-z_i^{2p}}$ and $\log\frac{k_{bi}^2}{k_{bi}^2-z_i^2} < \frac{z_i^2}{k_{bi}^2-z_i^2}$ $(p = 1)$.

Step 1: The LFC with constrained z_1 is given by

$$V_1 = V_{1,BLF} = \frac{1}{2}\log\frac{k_{b1}^2}{k_{b1}^2 - z_1^2}, \quad (1a)$$

The derivative of V_1

$$\dot{V}_1 = \frac{z_i}{k_{bi}^2 - z_i^2}(f_1 + g_1 x_2 - \dot{x}_{1d}),$$

$\dot{V} \le \gamma V$

Then the virtual control is

$$\alpha_1 = \frac{1}{g_1}[-f_1 + \dot{x}_{1d} - c_1 z_1]. \quad (2)$$

Then, substituting α_1 into $\dot{V}_1$ yields

$$\dot{V}_1 = -c_1\frac{z_1^2}{k_{b1}^2 - z_1^2} + \frac{g_1 z_1 z_2}{k_{b1}^2 - z_1^2} \le -c_1 \log\frac{k_{b1}^2}{k_{b1}^2 - z_1^2} + \frac{g_1 z_1 z_2}{k_{b1}^2 - z_1^2}. \quad (3)$$

Step n:

$$V_i = V_{i-1} + V_{i,QF}, \ i = 2, \cdots, n.$$
$$\dot{V}_n = -\sum c_i z_i^2$$
$$\alpha_2 = \frac{1}{g_2}\left(-c_2 z_2 - \frac{g_1 z_1}{k_{b1}^2 - z_1^2} + \cdots\right)$$

$\dot{V} \le -\gamma V + \delta$
Can be applied in more complex scenarios together with other methods.

2.5 Barrier Lyapunov function

Challenges

$$\alpha_2 = \frac{1}{g_2}\left(-c_2 z_2 - \frac{g_1 z_1}{k_{b1}^2 - z_1^2} + \cdots\right)$$

Understanding: Use very powerful input when z_1 approaches to its barrier. It is a **robustness-based** approach.

÷ Large control action may result when the states approach the boundary of the boundaries

÷ Upper and lower limits are assumed to be known

÷ The initial states have to stay in the constraints

Integral barrier Lyapunov functional (iBLF) relaxes the feasibility conditions

$$V_{i,iBLF} = \int_0^{z_i} \frac{\sigma k_{ai}^2}{k_{ai}^2 - (\sigma + \alpha_{i-1})^2} d\sigma$$

The properties of iBLF are

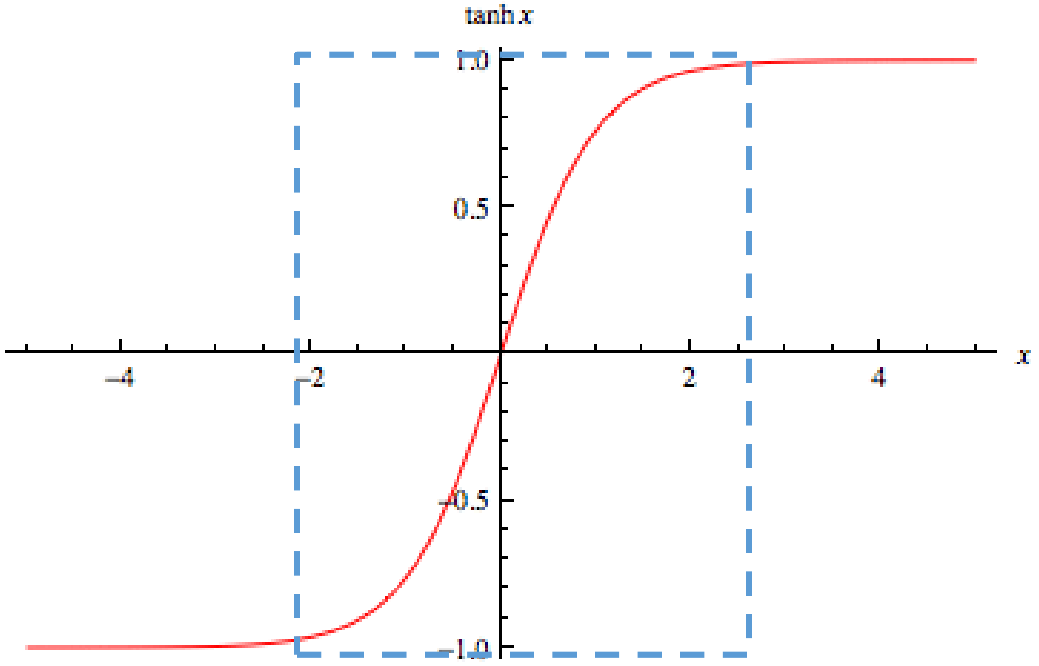
(i) $\frac{z_i^2}{2} \leq V_{i,iBLF} \leq z_i^2 \int_0^1 \frac{\beta k_{ai}^2}{k_{ai}^2 - (\beta z_i + \mathrm{sgn}(z_i) A_{i-1})^2} d\beta$

(ii) $\dot{V}_{i,iBLF} = \frac{k_{ai}^2 z_i}{k_{ai}^2 - x_i^2} \dot{z}_i + z_i \left(\frac{k_{ai}^2}{k_{ai}^2 - x_i^2} - \rho_i\right) \dot{x}_{id}$,

where $A_i < k_{ai}$ is the upper bound of $|x_{id}|$ and $\rho_i = \frac{k_{ai}}{2 z_i} \ln \frac{(k_{ai} + z_1 + x_{id})(k_{ai} - x_{id})}{(k_{ai} - z_i - x_{id})(k_{a1} + x_{id})}) \dot{x}_{id}$.

2.6 Hyperbolic tangent function (tanh)

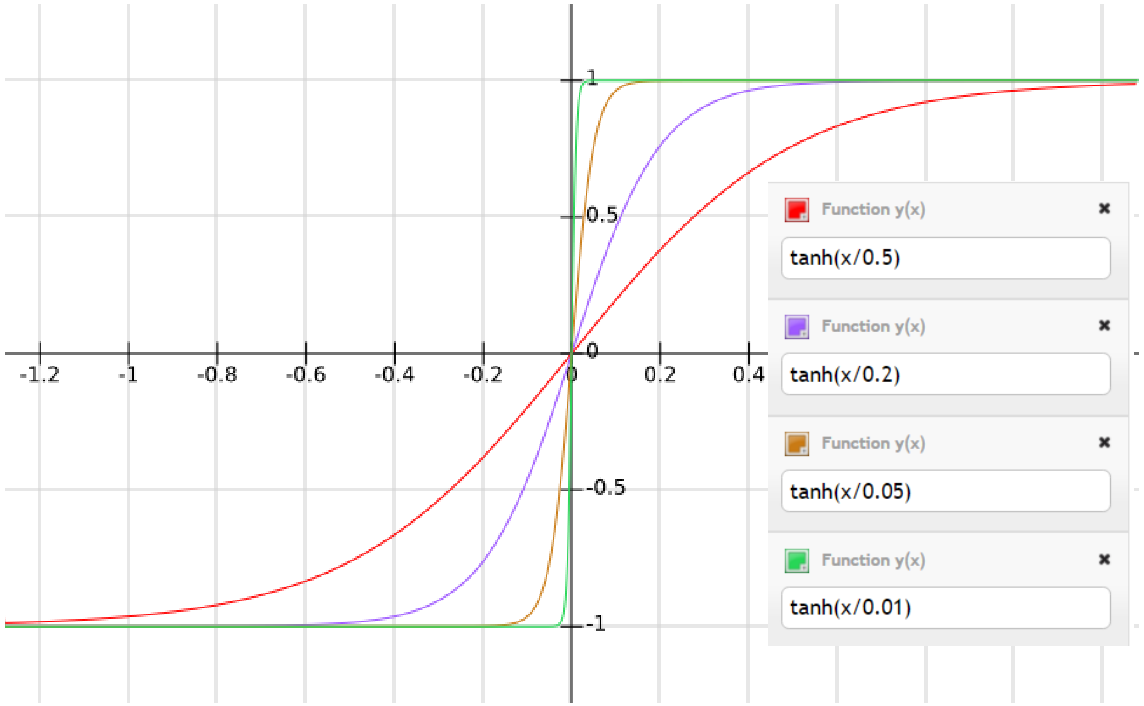
$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1}$$



Useful properties in control design:

- Odd function
- Smooth in $\mathbb{R}$
- $\tanh x \approx 1$ when $x \geq x_0$, $\tanh x \approx -1$ when $x \leq -x_0$
- $\tanh x \to 1$ when $x \to \infty$, $\tanh x \to -1$ when $x \to -\infty$
- Quick ramp from -1 to 1 near 0

$$\tanh\left(\frac{x}{\varepsilon}\right)$$



Useful properties in control design:

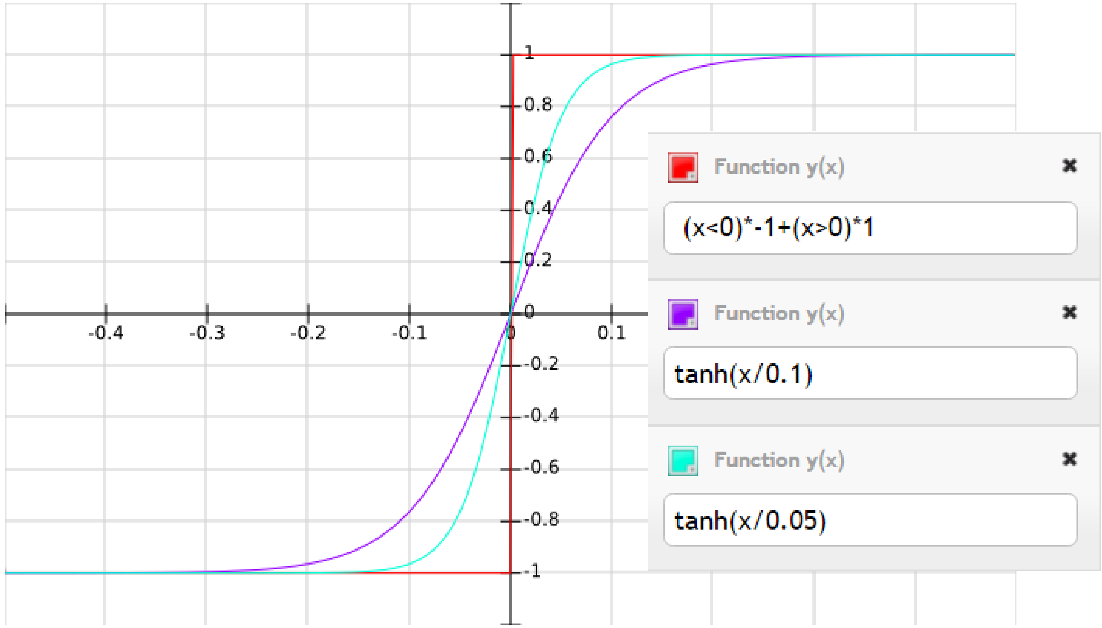
- The ramp becomes sharper with smaller ε

2.6 Hyperbolic tangent function (tanh)

Application 1: approximate of sign operator

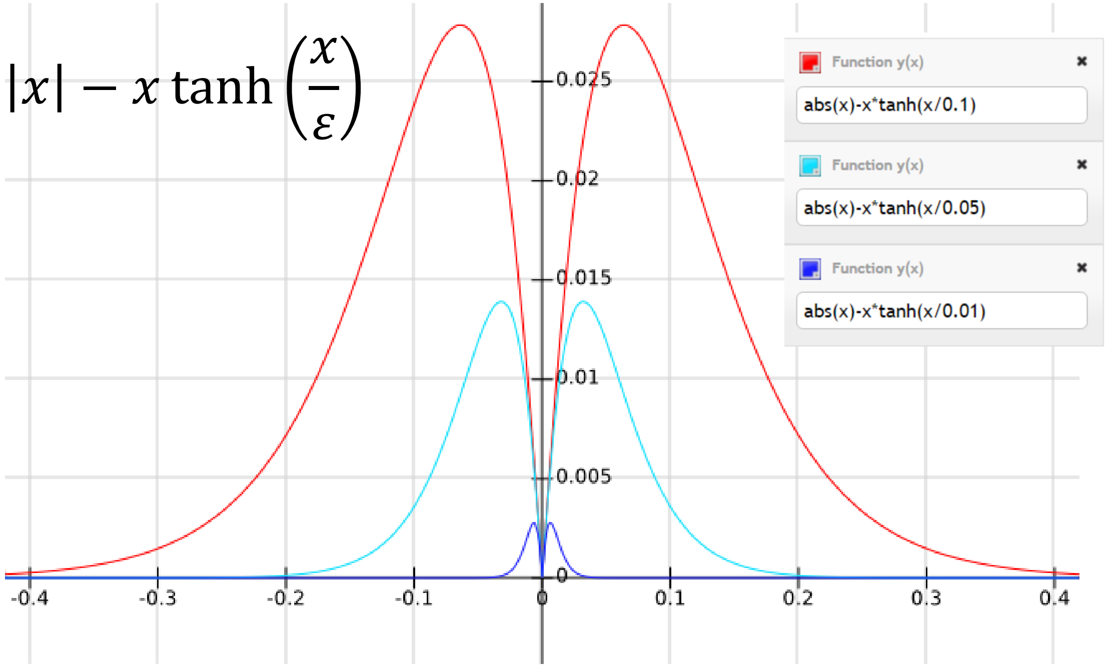
Approximation: $\operatorname{sgn} x \approx \tanh\left(\frac{x}{\varepsilon}\right)$

$|x| = \operatorname{sgn}(x)\, x = \tanh\left(\frac{x}{\varepsilon}\right) x$



Application 2: unknown disturbance

Important inequality: $0 \le |\eta| - \eta \tanh\left(\frac{\eta}{\varepsilon}\right) \le k_p \varepsilon,\ k_p = 0.2758$



Useful properties of $|x| - x\tanh\left(\frac{x}{\varepsilon}\right)$

- Maximum decreases with decreasing ε
- $|x| - x\tanh\left(\frac{x}{\varepsilon}\right) > 0, \forall x \in \mathbb{R}$

2.6 Hyperbolic tangent function (tanh)

Application 2: unknown disturbance

$$\dot{x}_i = f_i(\bar{x}_i) + g_i(\bar{x}_i)x_{i+1} + d_i, i \in \mathcal{I}$$
$$\dot{x}_n = f_n(\bar{x}_n) + g_n(\bar{x}_n)u + d_n,$$
$$y = x_1,$$

Control object: $x_1 - x_{1d} \to 0$ for $t \to \infty$;
Additional problem: Unknown disturbance d_i
Challenges:
- Difficult to cancel the term in $\dot{V}_i = \cdots + z_i d_i$ where d_i is unknown

Assumptions: The disturbance is bounded
- $|d_i| \le \bar{d}_i$
- $|d_i| \le \rho_i(\bar{x}_i)\theta_i$

Understanding: robustness ($-\tanh\left(\frac{z_1}{\varepsilon_1}\right)\rho\hat{\theta}$) + approximation ($\hat{\theta}$)

$$\dot{V}_1 \le -c_1 z_1^2 + \frac{|k_p\varepsilon_1\rho_1(x_1)|^2}{2} + \frac{|\theta_1|^2}{2} - \beta_i\tilde{\theta}_1\hat{\theta}_1 + z_1 g_1 z_2$$
$$= -c_1 z_1^2 + \frac{|k_p\varepsilon_1\rho_1(x_1)|^2}{2} + \frac{|\theta_1|^2}{2} - \beta_i\tilde{\theta}_1(\theta_1 - \tilde{\theta}_1) + z_1 g_1 z_2$$
$$= -c_1 z_1^2 - \frac{\beta_i}{2}\tilde{\theta}_1^2 + \frac{|k_p\varepsilon_1\rho_1(x_1)|^2}{2} + \frac{|\theta_1|^2}{2} + \frac{\beta_i}{2}\theta_1^2 + z_1 g_1 z_2$$
$$\dot{V}_1 \le -\gamma V_1 + \delta_1 + z_1 g_1 z_2$$

To Be Continued

Consider the Lyapunov function candidate

$$V_1 = V_{QF,1} + \frac{1}{2\gamma_1}\tilde{\theta}_1^2.$$

Then,

$$\dot{V}_1 = z_1[f_1 + g_1(\alpha_1 + z_2) + d_1 - \dot{x}_{1d}] - \frac{1}{\gamma_1}\tilde{\theta}_1\dot{\hat{\theta}}_1$$
$$\le z_1[f_1 + g_1(\alpha_1 + z_2) - \dot{x}_{1d}] + |z_1|\rho_1\theta_1 - \frac{1}{\gamma_1}\tilde{\theta}_1\dot{\hat{\theta}}_1$$
$$\le z_1[f_1 + g_1(\alpha_1 + z_2) - \dot{x}_{1d}] + [k_p\varepsilon_1 + z_1\tanh(\frac{z_1}{\varepsilon_1})]\rho_1\theta_1 - \frac{1}{\gamma_1}\tilde{\theta}_1\dot{\hat{\theta}}_1$$
$$\le z_1[f_1 + g_1(\alpha_1 + z_2) - \dot{x}_{1d}] + z_1\tanh(\frac{z_1}{\varepsilon_i})\rho_1\theta_1 + \frac{|k_p\varepsilon_1\rho_1(x_1)|^2}{2} + \frac{|\theta_1|^2}{2} - \frac{1}{\gamma_1}\tilde{\theta}_1\dot{\hat{\theta}}_1$$

The virtual controller and adaptive update law are given by

$$\alpha_1 = g_1^{-1}[-f_1 + \dot{x}_{1d} - c_1 z_1 - \tanh(\frac{z_1}{\varepsilon_1})\rho_1\hat{\theta}_1],$$
$$\dot{\hat{\theta}}_1 = z_1\tanh(\frac{z_1}{\varepsilon_1})\rho_1 - \gamma_1\beta_1\hat{\theta}_1.$$

2.6 Hyperbolic tangent function (tanh)

Application 3: Avoid singularity problem

If $\dot{V}_i = -\sum_{j=1}^{i-1} c_j z_j^2 + z_i\left(\ldots + g_i x_{i+1} + \frac{1}{z_i}\rho^2(z_i)\right)$ where$\rho(z_i)$ is an unknown class K function.

Challenge: Singularity problem of $\frac{\rho^2}{z_i}$ $\left(\lim_{z_i \to 0} \frac{\rho^2}{z_i} = \infty\right)$

Useful properties:

1. For any constant $\eta > 0$ and variable $z_i \in \mathbb{R}$, $\lim_{z_i \to 0} \frac{1}{z_i} \tanh^2\left(\frac{z_i}{\eta}\right) = 0$
2. If $|z_i| \geq 0.8814\eta$, $1 - 2\tanh^2\left(\frac{z}{\eta}\right) \leq 0$.

Solution:

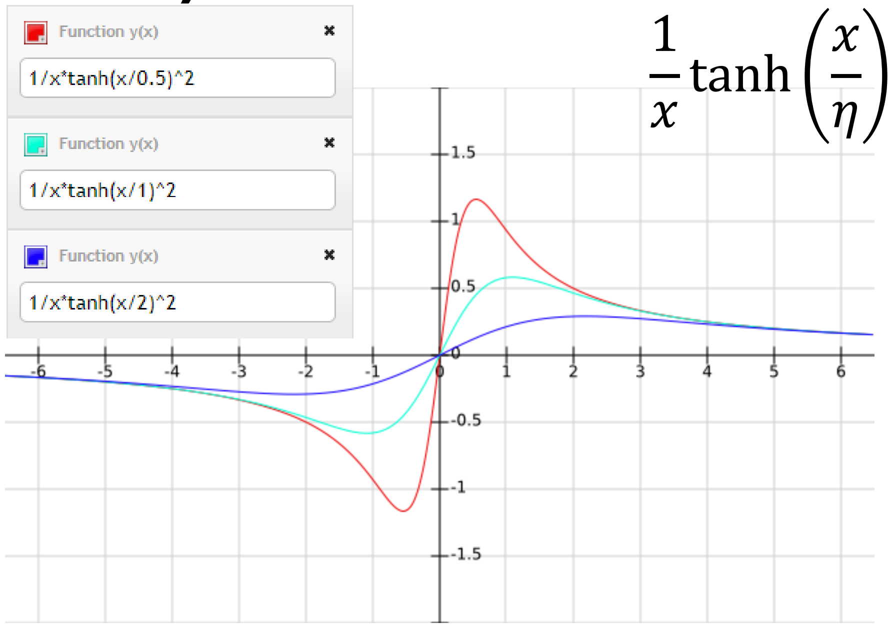
$$\dot{V}_i = -\sum_{j=1}^{i-1} c_j z_j^2 + z_i\left[\ldots + g_i x_{i+1} + \frac{1}{z_i}\rho^2(z_i)\left(1 + 2\tanh^2\left(\frac{z_i}{\eta}\right) - 2\tanh^2\left(\frac{z_i}{\eta}\right)\right)\right]$$

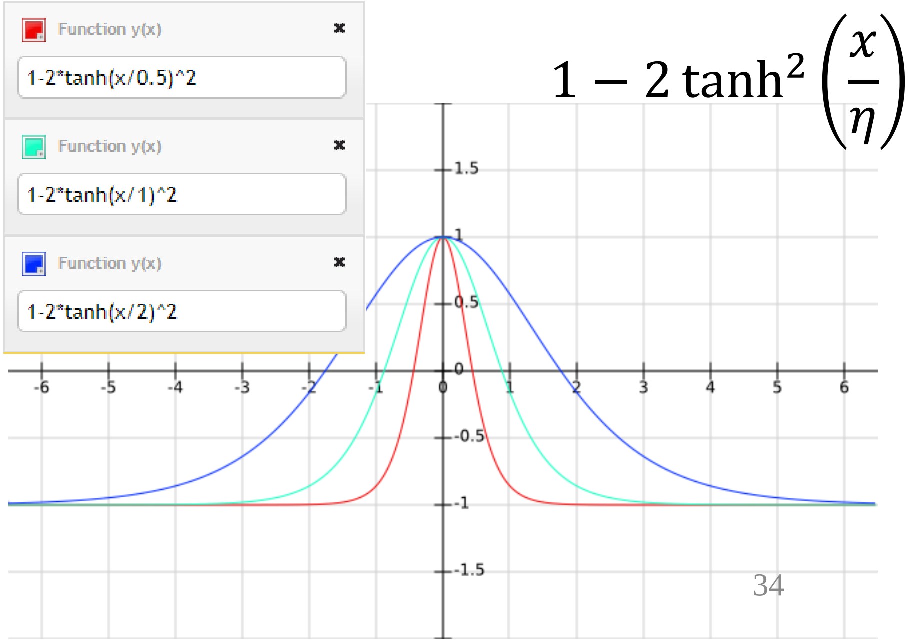
$$\leq -\sum_{j=1}^{i-1} c_j z_j^2 + z_i\left[\ldots + g_i x_{i+1} + 2\rho^2(z_i)\frac{1}{z_i}\tanh^2\left(\frac{z_i}{\eta}\right)\right]$$

$$+ \rho^2(z_i)\left(1 - 2\tanh^2\left(\frac{z_i}{\eta}\right)\right)$$

Then, $\left[\ldots + g_i x_{i+1} + 2\rho^2(z_i)\frac{1}{z_i}\tanh^2\left(\frac{z_i}{\eta}\right)\right]$ can be estimated.

- Case 1: If $z_i < 0.8814\eta$, $\mathrm{z_i}$ is bounded.
- Case 2: If $z_i \geq 0.8814\eta$, $\dot{\mathrm{V}} \leq -\gamma V + \delta$





Keywords when using these methods

- Assumptions
- Lyapunov function
- Stability criteria
- Key lemma (inequality)
- Robustness v.s. approximation

The main idea is to increase the control gains in the design to overcome all the uncanceled disturbance. However, these elegant methods give a guidance of how large the extra gain should be with a proved system stability.

3. Compare of robustness- and approximation-based methods

	Hardcore cancellation - robustness	Estimate and then cancel - observer
Too much		
Not enough		

Next lecture: preparation

How to convert significant amount of complex nonlinear systems into a form that the abovementioned methods can be applied.